Lecture 18

Remarks on Riemann Integral

Recall the Riemann integral of a function. $f: [a,b] \to \mathbb{R}$. Assume f is bounded and let $P: a = t_0 < t_1 < \dots < t_n = b$ be a partition of $[a,b]$. Form lower and upper sums

$$\overline{S}_P = \sum_{j=1}^{n} \underbrace{(\sup_{[t_{j-1},t_j]} f)}_{M_j} (t_j - t_{j-1}), \quad \underline{S}_P = \sum_{j=1}^{n} \underbrace{\inf_{[t_{j-1},t_j]} f}_{m_j} (t_j - t_{j-1})$$

Then form $\overline{I}_a^b = \inf_P \overline{S}_P$, $\underline{I}_a^b = \sup_P \underline{S}_P$

f is Riemann integrable if $\overline{I}_a^b = \underline{I}_a^b$ and the common number is the RI

$$\int_a^b f(x)\,dx.$$

Clearly, f cont. $\Rightarrow$ f R-integrable

Thm 1 Let $f: [a,b] \to \mathbb{R}$ be bounded.

(i) f R-integrable $\Rightarrow$ f L-integrable and $\int_a^b f(x)\,dx = \int_{[a,b]} f\,dm$.

(ii) f R-integrable $\Leftrightarrow$

$$m\left(\{x : f \text{ not cont. at } x\}\right) = 0.$$

Pf. (i) Recall that a partition P' is a refinement of P if P' contains all points in P (i.e. P' is obtained from P by subdividing some of the intervals $[t_{j-1}, t_j]$ further). A moments reflection (c.f. the construction of simple $\varphi_1 < \varphi_2 < \cdots$ tending pt.-wise to a measurable f) convinces you that $\overline{S}_{P'} \leq \overline{S}_P$ and $\underline{S}_{P'} \geq \underline{S}_P$. We may then take a sequence of partitions P_n, P_{n+1} a refinement of P_n and $\max_j t_j - t_{j-1} \to 0$, s.t.

$$\overline{I}_a^b = \lim_{n\to\infty} \overline{S}_{P_n}, \quad \underline{I}_a^b = \lim_{n\to\infty} \underline{S}_{P_n}$$

Let $\Phi_n = \sum_{j=1}^{m} M_j^n \chi_{(t_{j-1}, t_j)}$ and

$\varphi_n = \sum_{j=1}^{m} m_j^n \chi_{(t_{j-1}, t_j)}$, where

$P_n: a = t_0 < t_1 < \ldots < t_m = b$ and, as above, $M_j^n = \sup_{[t_{j-1}, t_j]} f$, $m_j^n = \inf_{[t_{j-1}, t_j]} f$.

Then $\overline{S}_{P_n} = \int_{[a,b]} \Phi_n \, dm$, $\underline{S}_{P_n} = \int_{[a,b]} \varphi_n \, dm$.

Since f is bounded, we have $m \le m_j^n \le M_j^n \le M$ for all n, j. As above we have $m \le \varphi_1 \le \varphi_2 \le \ldots \le M$ and $M \ge \Phi_1 \ge \Phi_2 \ge \ldots \ge m$. (WLOG: $m \le 0 \le M$). Thus, $\exists$ $\mathcal{B}_X$-meas. fcns $h \le g$ s.t. $\varphi_n \nearrow h$, $\Phi_n \searrow g$ and $h \le f \le g$ outside the countable set (null set) of points t_j^n in one of the P_n.

Since $|\varphi_n|, |\Phi_n| \leq \max(|m|, |M|)\chi_{[a,b]}$,

and RHS $\in L^1([a,b], m)$, DCT $\Rightarrow$

$$\underline{I}_a^b = \lim_{n\to\infty} \int \varphi_n \, dm = \int h \, dm$$

$$\overline{I}_a^b = \lim_{n\to\infty} \int \Phi_n \, dm = \int g \, dm$$

and since $\underline{I} = \overline{I} \Rightarrow \int \underbrace{(g-h)}_{\geq 0} \, dm = 0$

$\Rightarrow g = h$ m-a.e. $\Rightarrow h = f = g$ m-a.e.

$\Rightarrow f$ $\mathcal{L}$-meas. since m complete.

We also then have $f \in L^1$ and

$$\int_a^b f(x)\,dx = \underline{I}_a^b = \overline{I}_a^b = \int f \, dm.$$

as claimed.

(ii) Ex. 23 in Folland outlines a proof. DIY.

$\boxtimes$